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A Study of Solutions to the Aesthetic Field Equations

M. Muraskin

Department of Physics, University of North Dakota, Grand Forks, North Dakota

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We have found hundreds of solutions to the integrability equations in aesthetic field theory. The behavior of the solutions to the aesthetic field equations depends on which solution to the integrability equations we take. From computer runs down a coordinate axis we have found a type of solution where we have a maximum and a minimum, as well as the field going to zero at large distances along both directions. This kind of solution is quite prevalent. We call this type of solution a "pulse" solution. We have found the "pulse" solution in two and three dimensions as well as four dimensions. It appears regardless of whether certain symmetries are present or absent. We have taken a two- or three-dimensional $\Gamma^{\alpha}_{\beta}\gamma$ and made a four-dimensional theory from it with the use of a four-dimensional $e^{\alpha}i$. This process we call "imbedding." We have found imbedding has not affected the overall characteristics of the solution in the cases we considered. We were able to change the character of the solutions to some degree by altering the magnitude of some of the gammas—but this did not lead to solutions with significantly more wiggles. We also found an example of an oscillatory solution. The oscillations occurred in too regular a pattern to give a realistic model for basic behavior. However, this solution indicates that aesthetic field theory has more structure then we have ever seen before. We also obtained a solution in which errors took over so fast that the computer was literally helpless in telling us what is going on. In other solutions the field appears to increase without bounds. Whether this is due to singularities or to the presence of large numbers is not clear.

1. INTRODUCTION

We have been studying what we call Aesthetic Field Theory for some time (Muraskin, 1975). In this paper we discuss a set of mathematically aesthetic ideas and then demonstrate that these ideas define a field theory. We then show that these field equations have a nontrivial content. A two-particle scattering solution of the equations was studied in some detail

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(Muraskin and Ring, 1975a). Once we observe a scattering we are in a position to deduce force laws and hence we can compare our solution with real-world behavior. Our two-particle scattering solution, although nontrivial, appears too simple to be realistic. Thus a search has been initiated for more complicated solutions to the field equations.

In way of summary, we have required that all tensors and all orders of derivatives of the tensor fields be treated in a uniform way so far as their change is concerned. We work in a flat space–time described by Cartesian coordinates. We refer the reader to Muraskin (1975) for details.

The basic field equations are

$$\frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^m_{jk}\Gamma^i_{ml} - \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{jm}\Gamma^m_{kl} \equiv \Gamma^i_{jk;l} = 0 \tag{1.1}$$

The integrability equations associated with (1.1) are

$$\Gamma^i_{mk}R^m_{jpl} + \Gamma^i_{jm}R^m_{kpl} - \Gamma^m_{jk}R^i_{mpl} = 0 \tag{1.2}$$

with

$$R^t_{imk} \equiv \frac{\partial \Gamma^t_{ik}}{\partial x^l} - \frac{\partial \Gamma^t_{im}}{\partial x^k} - \Gamma^j_{im}\Gamma^t_{jk} + \Gamma^j_{ik}\Gamma^t_{jm} \tag{1.3}$$

Inserting (1.1) and (1.3) into (1.2) shows us that (1.2) is a set of algebraic nonlinear conditions. Once these conditions are satisfied at an arbitrary origin point they are satisfied everywhere as a consequence of the field equations. This gives us a handle on the accuracy of our computer solutions as we can test to see to how many decimal places (1.2) is satisfied after we have made a computer run away from the origin.

In four dimensions equation (1.2) represents a set of 384 algebraic conditions for the 64 Γ^i_{jk} (we have antisymmetry in the indices p and l). It is not clear at all that the equations (1.2) have any nontrivial solutions. However, by now, we have found a profusion of solutions to (1.2) and it is therefore our task to make a study of the solutions that have been obtained. Since (1.2) has nontrivial solutions we can say that solutions to (1.1) exist, locally at least, (Muraskin, 1975). Computer solutions suggest a global existence as well.

The equations written down above imply

$$T^{ij\cdots}{}_{kl\ldots;s} = 0 \tag{1.4}$$

where $T^{ij\cdots}{}_{kl\ldots}$ is any tensor function involving Γ^i_{jk} including any order derivative of tensor functions involving Γ^i_{jk}. ;s has the same mathematical structure as a covariant derivative. It should be remembered this correspondence is only formal as we are in a Cartesian system in a flat space.

As is pointed out in Eddington (1960) "A four dimensional continuum obeying Riemannian geometry can be represented graphically as a surface of four dimensions drawn in a Euclidean hyperspace of a sufficient number of dimensions." Thus in our theory, where we work in a flat space, we shall leave as a parameter the dimensions of space (Muraskin and Ring, 1975b, 1974).

No new algebraic conditions are needed in addition to (1.2) in order to establish local existence to the system (1.4).

When $T^{ij\cdots}{}_{kl\ldots}$ is taken to be Γ^i_{jk} (which we call the change function) we end up with the equations (1.1).

Thus there exists a set of fields, Γ^i_{jk}, such that products, contractions, and orders of derivatives of tensor combinations of the field are treated in a uniform way so far as their change is concerned.

The structure of (1.1) is such that if Γ^i_{jk} is given to us at one point then Γ^i_{jk} is determined at all points.

Although equations (1.1), (1.2), and (1.3) are conceptually simple the equations in practice are quite complicated and make analytic work difficult. For example, it would not be easy to reproduce the two-particle scattering solution given in Muraskin and Ring (1975a) by analytic means. We have found the computer to be an excellent tool in studying the solutions of (1.1), (1.2), and (1.3).

An extensive search for solutions of (1.1), (1.2), and (1.3) was begun in Muraskin and Ring (1976a). The techniques used are described in that article. In this paper we put together what has been learned from such a study.

2. INTEGRABILITY AND THE NATURE OF SOLUTIONS

We introduce $\Gamma^\alpha_{\beta\gamma}$ (as in Muraskin, 1975)

$$\Gamma^i_{jk} = e_\alpha{}^i e^\beta{}_j e^\gamma{}_k \Gamma^\alpha_{\beta\gamma} \tag{2.1}$$

If $\Gamma^\alpha_{\beta\gamma}$ obeys the integrability equations then Γ^i_{jk} will also.

The nature of the solutions to the field equation depend on the choice of $\Gamma^\alpha_{\beta\gamma}$. Examples to illustrate this point will be found below. We will only consider in this paper those $\Gamma^\alpha_{\beta\gamma}$ that satisfy the integrability equations.

(a) Γ^i_{jk} all equal leads analytically to a singular Γ^i_{jk}.

(b) The following data lead to Γ^i_{jk} being the same at all points. (All $\Gamma^\alpha_{\beta\gamma}$ are zero except for those listed below. This will be the same procedure we use in describing other solutions in this paper as well.) e^α_i in this paper will be taken to be the same e^α_i as in Muraskin and Ring (1975a). $\Gamma^\alpha_{\beta\gamma}$ is then taken to be

$$\begin{aligned}
&\Gamma^1_{11}=\Gamma^1_{22}=\Gamma^1_{33}=\Gamma^2_{11}=\Gamma^2_{22}=\Gamma^2_{33}=\Gamma^3_{11}=\Gamma^3_{22}=\Gamma^3_{33}=-0.1\\
&\Gamma^1_{12}=\Gamma^1_{23}=\Gamma^1_{31}=\Gamma^2_{12}=\Gamma^2_{23}=\Gamma^2_{31}=\Gamma^3_{12}=\Gamma^3_{23}=\Gamma^3_{31}=0.1
\end{aligned} \tag{2.2}$$

Unlike the trivial solution in Muraskin (1974) we do not have antisymmetry in any pairs of indices in the expression $g_{\alpha\beta}\Gamma^\alpha_{\beta\gamma}$ with $g_{\alpha\beta}=(1,1,1,1)$.

Another trivial solution occurs when we take

$$\begin{aligned}
&\Gamma^1_{11}=\Gamma^1_{12}=\Gamma^1_{33}=\Gamma^1_{30}=\Gamma^2_{23}=\Gamma^2_{20}=\Gamma^2_{01}=\Gamma^2_{02}\\
&\quad=\Gamma^3_{13}=\Gamma^3_{10}=\Gamma^3_{31}=\Gamma^3_{32}=\Gamma^0_{21}=\Gamma^0_{22}=\Gamma^0_{03}=\Gamma^0_{00}=0.1\\
&\Gamma^1_{23}=\Gamma^1_{20}=\Gamma^1_{01}=\Gamma^1_{02}=\Gamma^2_{11}=\Gamma^2_{12}=\Gamma^2_{33}=\Gamma^2_{30}=\Gamma^3_{21}=\Gamma^3_{22}\\
&\quad=\Gamma^3_{03}=\Gamma^3_{00}=\Gamma^0_{13}=\Gamma^0_{10}=\Gamma^0_{31}=\Gamma^0_{32}=-0.1
\end{aligned} \tag{2.3}$$

There is no way to avoid errors in a numerical calculation. The trivial solution (2.3) is much more sensitive to errors than the trivial solution (2.2). We see this in the integrability equations which are not as well satisfied as we move away from the origin.

(c) In Muraskin and Ring (1975a) we have described in detail a two-particle scattering solution. Computer pictures are to be found in this article. Running along the $\pm x$ axis we find one minimum and one maximum for Γ^1_{11}. Γ^1_{11} approaches zero eventually in both $+x$ and $-x$. All the components of Γ^i_{jk} are reasonably similar. This type of behavior along the $\pm x$ axis we will hereafter refer to as "pulse" behavior.

A difficulty with such a solution is that we only see structure around the origin. The fields go to zero outside the two-particle system. We would expect a more realistic situation to allow for multipulses along the axis. We cannot say from our computer studies that further structure does not develop when one goes far enough away from the particles. However, in our work we have found no evidence for this.

In Muraskin and Ring (1976a) we obtained such "pulse"-type solutions having varied properties. For example, some pulse solutions have $R^i_{jkl}=0$, some have $\Gamma^t_{tk}=\Gamma^t_{kt}$ etc. Sometimes we saw as many as four turnabout points along an axis (We attributed this to an "arm" structure in Muraskin and Ring (1975a)).

However, all our "pulse" solutions found previously had the property that $\Gamma^\alpha_{\beta\gamma}$ is symmetric under the following interchange of indices:

$$1\to2 \quad 2\to3 \quad 3\to1 \tag{2.4}$$

or

$$1 \rightarrow 2 \quad 2 \rightarrow 3 \quad 3 \rightarrow 0 \quad 0 \rightarrow 1 \tag{2.5}$$

This latter situation is discussed in Muraskin and Ring (1976b).

We have now found that pulse solutions appear without starting off with such symmetries. An example is

$$\begin{aligned} \Gamma^1_{21} &= \Gamma^1_{22} = \Gamma^1_{23} = \Gamma^2_{11} = \Gamma^2_{12} = \Gamma^2_{13} = \Gamma^3_{31} = \Gamma^3_{32} = \Gamma^3_{33} = \Gamma^0_{10} = \Gamma^0_{20} \\ &= \Gamma^0_{30} = \Gamma^0_{00} = \Gamma^0_{01} = \Gamma^0_{02} = \Gamma^0_{03} = 0.1 \\ \Gamma^1_{00} &= \Gamma^2_{00} = \Gamma^3_{00} = -0.1 \end{aligned} \tag{2.6}$$

Without the symmetry it is not clear that runs along $\pm y, \pm z, \pm t$ should show features similar to the runs along $\pm x$. We therefore ran along all the coordinate axes and found the same pattern in each case. Also selected maps showed no unusual behavior from what we had seen before. We have found a fair number of solutions not starting with the symmetries (2.4) or (2.5). We note that once $e^\alpha{}_i$ transforms a solution the symmetry (2.4) or (2.5) would be lost. We have not proved that there is no $e^\alpha{}_i$ transformation in all these instances that would lead to the symmetry (2.4) or (2.5). All we can say is that we do not know of any $e^\alpha{}_i$ transformation that would unmask such a symmetry.

The conclusion we reach is that pulse solutions are easily obtained. Solutions having such behavior have varied mathematical properties.

(d) We have found a large number of solutions where the field component we are studying (which we have taken to be Γ^1_{11}) appears to get bigger and bigger in magnitude with no bound. Before this happens there may be one, two, or (rarely) three turnabout points. In many instances there are no turnabouts points. Often in this latter situation we have a bound of zero in either the $+x$ or $-x$ direction.

It is not clear whether a singularity is developing or we are dealing with extremely large numbers.

An example of Γ^1_{11} getting bigger and bigger in the $-x$ direction but going to zero in the $+x$ direction with no turnabout points is

$$\begin{aligned} \Gamma^1_{11} &= \Gamma^1_{12} = \Gamma^1_{13} = \Gamma^1_{03} = \Gamma^2_{21} = \Gamma^2_{22} = \Gamma^2_{23} = \Gamma^2_{01} = \Gamma^3_{31} = \Gamma^3_{32} \\ &= \Gamma^3_{02} = \Gamma^3_{33} = \Gamma^0_{01} = \Gamma^0_{02} = \Gamma^0_{03} = -0.1 \\ \Gamma^1_{01} &= \Gamma^1_{30} = \Gamma^2_{10} = \Gamma^2_{02} = \Gamma^3_{20} = \Gamma^3_{03} = \Gamma^0_{00} = +0.1 \end{aligned} \tag{2.7}$$

An example of Γ^1_{11} getting bigger and bigger in the $-x$ direction with three turnabout points and with Γ^1_{11} approaching zero in the $+x$ direction is

$$\begin{aligned}&\Gamma^1_{11}=\Gamma^1_{22}=\Gamma^1_{00}=\Gamma^2_{12}=\Gamma^2_{21}=\Gamma^2_{00}=\Gamma^3_{33}=\Gamma^0_{11}=\Gamma^0_{12}=\Gamma^0_{21}=\Gamma^0_{22}=-0.1\\&\Gamma^1_{12}=\Gamma^1_{10}=\Gamma^1_{21}=\Gamma^1_{20}=\Gamma^1_{01}=\Gamma^1_{02}=\Gamma^2_{11}=\Gamma^2_{22}=\Gamma^2_{10}\\&\quad=\Gamma^2_{20}=\Gamma^2_{01}=\Gamma^2_{02}=\Gamma^0_{10}=\Gamma^0_{20}=\Gamma^0_{00}=\Gamma^0_{01}=\Gamma^0_{02}=0.1\end{aligned} \tag{2.8}$$

(e) We have found an oscillatory solution. We take $\Gamma^\alpha_{\beta\gamma}$ to be

$$\begin{array}{llll}
\Gamma^1_{11}=0.1 & \Gamma^1_{12}=0.1 & \Gamma^1_{13}=0 & \Gamma^1_{10}=0\\
\Gamma^1_{21}=-0.1 & \Gamma^1_{22}=-0.1 & \Gamma^1_{23}=0.1 & \Gamma^1_{20}=0.1\\
\Gamma^1_{31}=-0.1 & \Gamma^1_{32}=-0.1 & \Gamma^1_{33}=0.1 & \Gamma^1_{30}=0.1\\
\Gamma^1_{01}=0 & \Gamma^1_{02}=0 & \Gamma^1_{03}=-0.1 & \Gamma^1_{00}=-0.1\\
\Gamma^2_{11}=0 & \Gamma^2_{12}=0 & \Gamma^2_{13}=-0.1 & \Gamma^2_{10}=-0.1\\
\Gamma^2_{21}=0 & \Gamma^2_{22}=0 & \Gamma^2_{23}=0 & \Gamma^2_{20}=0\\
\Gamma^2_{31}=0 & \Gamma^2_{32}=0 & \Gamma^2_{33}=0 & \Gamma^2_{30}=0\\
\Gamma^2_{01}=0.1 & \Gamma^2_{02}=0.1 & \Gamma^2_{03}=0 & \Gamma^2_{00}=0\\
\Gamma^3_{11}=0.1 & \Gamma^3_{12}=0.1 & \Gamma^3_{13}=0 & \Gamma^3_{10}=0\\
\Gamma^3_{21}=0 & \Gamma^3_{22}=0 & \Gamma^3_{23}=0 & \Gamma^3_{20}=0\\
\Gamma^3_{31}=0 & \Gamma^3_{32}=0 & \Gamma^3_{33}=0 & \Gamma^3_{30}=0\\
\Gamma^3_{01}=0 & \Gamma^3_{02}=0 & \Gamma^3_{03}=-0.1 & \Gamma^3_{00}=-0.1\\
\Gamma^0_{11}=0 & \Gamma^0_{12}=0 & \Gamma^0_{13}=-0.1 & \Gamma^0_{10}=-0.1\\
\Gamma^0_{21}=-0.1 & \Gamma^0_{22}=-0.1 & \Gamma^0_{23}=0.1 & \Gamma^0_{20}=0.1\\
\Gamma^0_{31}=-0.1 & \Gamma^0_{32}=-0.1 & \Gamma^0_{33}=0.1 & \Gamma^0_{30}=0.1\\
\Gamma^0_{01}=0.1 & \Gamma^0_{02}=0.1 & \Gamma^0_{03}=0 & \Gamma^0_{00}=0
\end{array} \tag{2.9}$$

We found oscillatory behavior along all coordinate axes for Γ^1_{11}. We list in Table I the turnabout points along the y axis near the origin.

TABLE I. Turnabout Points along the y Axis

y coordinate at turnabout	Value of Γ^1_{11}
43	1.500
23	−0.300
3	0.249
−17	0.081
−37	0.133
−57	0.117
−77	0.122
−97	0.1203
−117	0.1207
−137	0.12059
−157	0.12063
−177	0.120622
−197	0.120626

The turnabout points are equally spaced along the axes. It appears from the regularity that the number of turnabout points may well be infinite. The amplitude of the oscillations decreases as we move down Table I. Around $y = -197$ the change of amplitude becomes so small as to be barely detectable.

In our previous work we never found any solution obeying integrability that has more than four turnabout points along an axis. Thus with 13 observed turnabout points in Table I it is clear that the aesthetic field equations are capable of new effects not previously seen.

Our present solution is too regular to be a realistic model for particles. Also the contour lines do not appear to close as far as we could tell. It is not determined whether the oscillations are bounded.

This solution has a completely different character from those seen previously. This solution was also rather difficult to find. In contrast the other kinds of solutions we have described appeared in great numbers.

(f) There exist solutions we have found for which the computer appears helpless in determining what is going on. Consider the following $\Gamma^\alpha_{\beta\gamma}$:

$$\begin{aligned}\Gamma^1_{13}&=\Gamma^1_{02}=\Gamma^1_{31}=\Gamma^1_{20}=\Gamma^2_{11}=\Gamma^2_{22}=\Gamma^2_{33}=\Gamma^2_{00}\\&=\Gamma^3_{13}=\Gamma^3_{20}=\Gamma^3_{31}=\Gamma^3_{02}=\Gamma^0_{11}=\Gamma^0_{22}=\Gamma^0_{33}=\Gamma^0_{00}=0.1\\\Gamma^1_{10}&=\Gamma^1_{21}=\Gamma^1_{32}=\Gamma^1_{03}=\Gamma^2_{10}=\Gamma^2_{21}=\Gamma^2_{32}=\Gamma^2_{03}=\Gamma^3_{10}=\Gamma^3_{21}\\&=\Gamma^3_{32}=\Gamma^3_{03}=\Gamma^0_{10}=\Gamma^0_{21}=\Gamma^0_{32}=\Gamma^0_{03}=-0.1\end{aligned} \tag{2.10}$$

Errors become important as one runs down the axes as reflected from tests involving the integrability equations. Normally one lowers the grid to get better accuracy. In this case it does not work. The bigger grid gives greater accuracy as long as the grid is not too big to start. When one lowers the grid there are more points to calculate to get to the same position which brings in errors. In our previous work this effect did not dominate. In the case above it also appears that the structure of the solution is such that one is often adding big numbers to little ones, which is another source of errors. At any rate, the accuracy diminishes so quickly that we have no idea what the attributes of the solution are. It may be that something interesting is going on—but we just cannot say. We have found other sets of data as well for which the computer fails us.

3. TWO DIMENSIONS; IMBEDDING IN HIGHER-DIMENSIONAL THEORY; MAGNITUDE EFFECT

Lower dimensions does not preclude greater structure far from the origin. This, we remember, was a drawback of the pulse solutions. This suggests that a study of lower dimensions may be useful. Lower dimensions are more easy to work with. We work here in two dimensions.

The $R^i_{jkl}=0$ integrability equations collapse into four equations for eight unknowns. We then have the freedom of assigning four of the gammas arbitrarily and we use the integrability equations to solve for the other four. We have in this way obtained many solutions. Interestingly the pulse solution appears quite often. An example of data leading to a pulse solution is

$$\begin{aligned} &\Gamma^1_{11}=0, \quad \Gamma^1_{12}=0.1, \quad \Gamma^2_{11}=-0.1, \quad \Gamma^2_{21}=0 \\ &\Gamma^1_{21}=0.1, \quad \Gamma^1_{22}=0, \quad \Gamma^2_{12}=0, \quad \Gamma^2_{22}=0.1 \end{aligned} \tag{3.1}$$

We take $e^\alpha{}_i$ to be

$$\begin{aligned} &e^1_1=0.88 \quad e^1_2=-0.42 \\ &e^2_1=0.5 \quad e^2_2=0.9 \end{aligned} \tag{3.2}$$

The theory is then purely two dimensional. Another possibility is to take $\Gamma^\alpha_{\beta\gamma}$ to be as above but then take $e^\alpha{}_i$ to be the four dimensional ones used in Muraskin and Ring (1975a, page 515). Using (2.1) we then get a theory involving 64 Γ^i_{jk}. We call this procedure "imbedding" of a two-dimensional $\Gamma^\alpha_{\beta\gamma}$ in a four-dimensional theory.

We find that using a four-dimensional $e^\alpha{}_i$ does not alter the qualitative character of the pulse solution.

Imbedding the $\Gamma^\alpha_{\beta\gamma}$ given above in a three-dimensional theory also gives the same type of behavior as in a pure two-dimensional theory (at least for the pulse solution).

Getting a bit ahead of ourselves, the imbedding idea gives us a chance to take a three-dimensional $\Gamma^\alpha_{\beta\gamma}$ and extend the theory to four dimensions using a four-dimensional $e^\alpha{}_i$. Then the space coordinates would be treated differently from the time coordinates as far as $\Gamma^\alpha_{\beta\gamma}$ is concerned. Thus the imbedding idea gives us a mathematical way to treat time different from space.

Another nice feature of two-dimensional solutions is that we can change the magnitude of some of the gammas readily and still have a solution. Does the magnitude of some of the gammas affect the character of the solution?

To some degree we know from four-dimensional work that the change of magnitude of some of the gammas does affect the character of solutions. Consider in four dimensions the following $\Gamma^\alpha_{\beta\gamma}$:

$$\begin{aligned}
&\Gamma^1_{10}=\Gamma^2_{20}=\Gamma^3_{30}=\Gamma^0_{00}=\Gamma^1_{01}=\Gamma^2_{02}=\Gamma^3_{03}=0.1\\
&\Gamma^1_{23}=\Gamma^2_{31}=\Gamma^3_{12}=-0.1\\
&\Gamma^1_{32}=\Gamma^2_{13}=\Gamma^3_{21}=0.1\\
&\Gamma^0_{11}=\Gamma^0_{22}=\Gamma^0_{33}=\psi
\end{aligned}\tag{3.3}$$

No matter what choice of ψ we take, we get a solution to the integrability equations (1.2). As representative we study Γ^1_{11} as we have been doing thus far. For ψ negative we get a pulse solution. We have chosen ψ to be $-0.01, -0.1, -1.0, -10.0, -100.0$. On the other hand when ψ is positive the solution approaches 0 in $-x$ but appears to grow indefinitely in $+x$ with no turnabout points in between. When the magnitude of ψ is changed we appear to get nothing intermediate between the pulse solution and the solution in which Γ^1_{11} gets bigger and bigger with no turnaround points.

Let us go back to the two-dimensional case. We allow Γ^1_{21} to be a parameter. $\Gamma^1_{12}, \Gamma^2_{21}, \Gamma^2_{12}$ were assigned the same values throughout. We obtained the following solutions of integrability:

$$\begin{array}{llll}
\multicolumn{2}{c}{\text{(a)}} & \multicolumn{2}{c}{\text{(b)}} \\
\Gamma^1_{11}=1.0 & \Gamma^1_{12}=0.5 & \Gamma^1_{11}=1.0 & \Gamma^1_{12}=0.5 \\
\Gamma^1_{21}=5.0 & \Gamma^1_{22}=-237.5 & \Gamma^1_{21}=-5.0 & \Gamma^1_{22}=-262.5 \\
\Gamma^2_{11}=0.016 & \Gamma^2_{12}=0.2 & \Gamma^2_{11}=-0.016 & \Gamma^2_{12}=0.2 \\
\Gamma^2_{21}=0.8 & \Gamma^2_{22}=-14.0 & \Gamma^2_{21}=0.8 & \Gamma^2_{22}=16.0 \\
\multicolumn{2}{c}{\text{(c)}} & \multicolumn{2}{c}{\text{(d)}} \\
\Gamma^1_{11}=1.0 & \Gamma^1_{12}=0.5 & \Gamma^1_{11}=1.0 & \Gamma^1_{12}=0.5 \\
\Gamma^1_{21}=-0.7 & \Gamma^1_{22}=-6.65 & \Gamma^1_{21}=-1.5 & \Gamma^1_{22}=-26.25 \\
\Gamma^2_{11}=-.1142857 & \Gamma^2_{12}=0.2 & \Gamma^2_{11}=-.0533333 & \Gamma^2_{12}=0.2 \\
\Gamma^2_{21}=0.8 & \Gamma^2_{22}=3.1 & \Gamma^2_{21}=0.8 & \Gamma^2_{22}=5.5
\end{array} \tag{3.4}$$

Changing Γ^1_{21} as above did not change the pulse character of the solution. The magnitude of Γ^1_{11} at the maximum (minimum) was changed. The location of the maximum and minimum was not very different.

However, when Γ^1_{21} took on the value of 0.7 we got

$$\begin{array}{ll}
\Gamma^1_{11}=1.0 & \Gamma^1_{12}=0.5 \\
\Gamma^1_{21}=0.7 & \Gamma^1_{22}=-3.15 \\
\Gamma^2_{11}=0.1142857 & \Gamma^2_{12}=0.2 \\
\Gamma^2_{21}=0.8 & \Gamma^2_{22}=-1.1
\end{array} \tag{3.5}$$

Here Γ^1_{11} approached zero in the $-x$ but took off in the $+x$ direction. The situation is similar to (3.3). When we took $\Gamma^1_{21}=0.01$ we got for a solution[1]

$$\begin{array}{ll}
\Gamma^1_{11}=1.424490 & \Gamma^1_{12}=0.5 \\
\Gamma^1_{21}=0.01 & \Gamma^1_{22}=-0.533555 \\
\Gamma^2_{11}=-0.749688 & \Gamma^2_{12}=0.2 \\
\Gamma^2_{21}=0.8 & \Gamma^2_{22}=1.316666
\end{array} \tag{3.6}$$

[1]The integrability equation for Γ^1_{11} is a cubic. One of the solutions here has $\Gamma^1_{11}=1.0$ but we listed above another solution (3.6).

Here Γ^1_{11} again appeared to grow without bound but in this case there was a turning point for Γ^1_{11}. Γ^1_{11} approached 0 in the $-x$ direction.

Thus we see that the change of magnitude of Γ^1_{21} appears to affect the character of solutions to some degree. Hard and fast conclusions cannot be reached since it is not clear whether we are dealing with unboundedness or large numbers. The magnitude effect in these instances did not lead to any greater number of wiggles (turnabout points).

We see that the kinds of things we see along the axes in two dimensions also appear in four dimensions. On the other hand we have not obtained an oscillatory solution in two (or three) dimensions. Whether this is due to the difficulty of finding such a solution or not, we cannot say.

4. THREE DIMENSIONS

The pulse solution appears in three dimensions as well. This occurs with or without imbedding (using a four-dimensional $e^{\alpha}{}_{i}$). An example of a three-dimensional pulse solution is

$$\begin{aligned}&\Gamma^1_{11}=\Gamma^1_{13}=\Gamma^1_{32}=\Gamma^2_{12}=\Gamma^2_{21}=\Gamma^2_{22}=\Gamma^2_{23}=\Gamma^2_{32}=\Gamma^3_{12}=\Gamma^3_{31}=\Gamma^3_{33}=0.1\\&\Gamma^1_{12}=\Gamma^1_{22}=\Gamma^3_{22}=\Gamma^3_{32}=-0.1\end{aligned}\tag{4.1}$$

We have mapped this solution and found a planar maximum and minimum so typical of the pulse solutions. Another example of a pulse solution is

$$\begin{aligned}&\Gamma^1_{13}=\Gamma^1_{31}=\Gamma^2_{23}=\Gamma^2_{32}=\Gamma^3_{12}=\Gamma^3_{33}=0.1\\&\Gamma^1_{11}=\Gamma^1_{21}=\Gamma^2_{21}=\Gamma^2_{22}=\Gamma^3_{22}=\Gamma^3_{11}=-0.1\end{aligned}\tag{4.2}$$

We also found in three dimensions solutions that approach 0 in one axis direction but seem to grow without bounds in the opposite direction. We found some sets of data that have a few turnabout points and then seem to grow without bounds. In short the three-dimensional solutions were similar along the axes and in planar maps to the two- and four-dimensional work. We have not as yet found an oscillatory solution in three dimensions. Symmetries like $\Gamma^i_{jk}=\Gamma^i_{kj}$ were not a factor in whether pulse solutions appeared or not.

5. ADDITIONAL WORK WITH THE OSCILLATORY SOLUTION

We imbedded the oscillatory solution in a five-dimensional theory using a five dimensional $e^{\alpha}{}_{i}$. We still got oscillatory behavior with the oscillations spaced at regular intervals along the axis. Thus the imbedding did not alter the characteristics of the solution.

6. SUMMARY

By now we have studied hundreds of solutions to the integrability equations.

We have found similar type behavior in two, three, and four dimensions when we look at axes runs and some maps (except for the oscillatory solution, which we found only in four dimensions). This leaves open in our mind why four dimensions seems to be preferred in the actual world. The pulse type behavior was obtained quite readily. The pulse solution appears to satisfy the natural boundary conditions $\Gamma^i_{jk} \to 0$ at infinity as inferred from our computer work. Imbedding in a higher-dimensional theory did not alter the characteristics of the solution in the many cases we studied. Changing the magnitude of some of the components has some effect on the character of the solution but did not enable us to obtain significantly more wiggles.

In four dimensions we found an oscillatory solution. Thus aesthetic field theory has solutions with greater structure than had been obtained previously.

Is there an intermediary type solution between the bounded pulse solution and the solution in which the gammas appear unbounded after only a small number of turnabout points? The oscillatory solution may suggest that there is—but we have not yet found it.

We have found a solution for which the computer appears almost helpless. We have no idea as to the behavior of such a solution.

We have thus far analyzed a great number of solutions of the integrability equation equations. The character of the solutions to the field equations depends on which solutions of the integrability equations we study. As the integrability equations are so complicated we do not have a complete understanding of the information buried in aesthetic field theory.

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